$$\begin{aligned} &= \frac{-\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} \qquad \text{(by (e))} \\ &= \frac{-1}{1+\sqrt{1+\frac{1}{n}}} \\ &\to \frac{-1}{1+1} = -\frac{1}{2}. \end{aligned}$$

2. Let $(x_n)_{n\in\mathbb{N}} = \left((-1)^{n+1}\right)_{n\in\mathbb{N}} = (1,-1,1,-1,\ldots)$ and let $(y_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}} = (-1,1,-1,1,\ldots)$. $\forall n \in N$, $x_n + y_n = 0$, $x_n \cdot y_n = -1 = \dfrac{x_n}{y_n}$.

3. Write $x_n = \dfrac{(x_n+y_n)+(x_n-y_n)}{2}$ and $y_n = \dfrac{(x_n+y_n)-(x_n-y_n)}{2}$. By Theorem 3.2 both $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge.

4. $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$.

5. $\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n} \forall n$. Since $\pm\dfrac{1}{n} \to 0$, $\dfrac{\cos n}{n} \to 0$ by the Squeeze Theorem.

6. Let $B > 0$ with $|x_n| \le B \ \forall\, n \in \mathbb{N}$. Then $0 \le |x_n y_n| \le B\,|y_n| \to 0$. By the Squeeze Theorem, $|x_n y_n| \to 0$ and so $x_n y_n \to 0$.

7. (a) Let $x_n = (-1)^n$ and $y_n = 1 \ \forall n \in \mathbb{N}$. Then $(x_n y_n)_{n\in\mathbb{N}} = (x_n)_{n\in\mathbb{N}}$ does not converge.

   (b) Let $x_n = n^2$ and $y_n = \frac{1}{n} \ \forall n \in \mathbb{N}$. Then $(x_n y_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$ is unbounded and hence does not converge.

8. Let $x \in \mathbb{R}$. Since the irrational numbers are dense in $\mathbb{R}$, $\forall n \in \mathbb{N}$ choose an irrational $x_n \in (x - \frac{1}{n}, x + \frac{1}{n})$. Then $-\dfrac{1}{n} < x_n - x < \dfrac{1}{n} \ \forall n \in \mathbb{N}$. By the Squeeze Theorem, $x_n - x \to 0$ or, equivalently, $x_n \to x$.

3.3 Subsequences

1. (a) $(1,2,1,3,1,4,\ldots)$ has the constant sequence $(1,1,1,\ldots)$ as a subsequence.

(b) $(1, 2, 3, 4, \ldots)$. Every subsequence is unbounded because the ordering of a subsequence must be the same as the ordering of the sequence. From Theorem 3.15 of Section 3.7, every subsequence must have limit ∞.

(c) No. See Section 5 of this chapter. In our opinion it is sometimes worthwhile to have students think about ideas before you cover them.

2. (a) e. This is a subsequence of $\left((1+\frac{1}{n})^n\right)_{n\in\mathbb{N}}$.

(b) $$\left(1+\frac{1}{2n}\right)^n = \left[\left(1+\frac{1}{2n}\right)^{2n}\right]^{\frac{1}{2}} \to e^{\frac{1}{2}} \text{ since } \left(\left(1+\frac{1}{2n}\right)^{2n}\right)_{n\in\mathbb{N}}$$
is a subsequence of $\left(\left(1+\frac{1}{n}\right)^n\right)_{n\in\mathbb{N}}$.

3. Draw $y = \sin x$ on $[0, 2\pi]$ and note where $\sin x = \pm\frac{1}{2}$. $\forall k \in \mathbb{N}$, choose one integer $n_k \in \left(\frac{\pi}{6} + 2(k-1)\pi, \frac{5\pi}{6} + 2(k-1)\pi\right)$ and one integer $m_k \in \left(\frac{7\pi}{6} + 2(k-1)\pi, \frac{11\pi}{6} + 2(k-1)\pi\right)$. Then $(\sin n_k)_{k\in\mathbb{N}}$ and $(\sin m_k)_{k\in\mathbb{N}}$ are subsequences of $(\sin n)_{n\in\mathbb{N}}$, and $\sin n_k \geq \frac{1}{2}$ and $\sin m_k \leq -\frac{1}{2}$ $\forall k$. By Theorem 3.6, $(\sin n)_{n\in\mathbb{N}}$ cannot converge.

4. First note that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are both subsequences of $(z_n)_{n\in\mathbb{N}}$. By Theorem 3.6, if $z_n \to x$, then $x_n \to x$ and $y_n \to x$. Now suppose $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x \in \mathbb{R}$, and note that
$$z_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ y_{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$$
Let U be a neighborhood of x. Choose n_1 and n_2 in $\mathbb{N}$ such that $n \geq n_1 \Rightarrow x_n \in U$ and $n \geq n_2 \Rightarrow y_n \in U$. Let $n_0 = \max\{2n_1 - 1, 2n_2\}$. Then $n \geq n_0 \Rightarrow z_n \in U$, and so $z_n \to x$.

5. (a) First note that $(x_n)_{n\in\mathbb{N}}$ is unbounded above $\Leftrightarrow \forall B > 0, \exists\, n \in \mathbb{N}$ such that $x_n > B$. For $B = 1$, let n_1 be the first positive integer with $x_{n_1} > 1$. For $B = \max\{x_{n_1}, 2\}$, let n_2 be the first positive integer greater than n_1 with $x_{n_2} > B$, and so $x_{n_2} > 2$. (If no such $n_2 > n_1$ exists, then $(x_n)_{n\in\mathbb{N}}$ would be bounded above by this B.) For $B = \max\{x_{n_2}, 3\}$, let n_3 be the first positive integer greater than n_2 with $x_{n_3} > B$, and so $x_{n_3} > 3$. Continue by induction, getting $x_{n_k} > k$ at each stage.

(b) Since $(x_n)_{n\in\mathbb{N}}$ is unbounded below, $(-x_n)_{n\in\mathbb{N}}$ is unbounded above. By part (a), $\exists$ a subsequence $(-x_{n_k})_{k=1}^{\infty}$ of $(-x_n)_{n\in\mathbb{N}}$ such that $-x_{n_k} > k \ \forall k \in \mathbb{N}$. Then $(x_{n_k})_{k=1}^{\infty}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ and $x_{n_k} < -k \ \forall k \in \mathbb{N}$.

3.4 Monotone Sequences

It is worthwhile to point out to students that sometimes the proof of a theorem yields more than is actually stated in the theorem. Case in point is Theorem 3.7, the Monotone Convergence Theorem. The statement of the theorem is that a bounded monotone sequence converges. The proof not only demonstrates that such a sequence converges, but it shows us what the value of the limit must be.

1. The first three terms are $1, \frac{5}{4}, \frac{11}{8}$. Claim: $x_n < 2 \ \forall n \in \mathbb{N}$.
   If $x_k < 2$, then

$$x_{k+1} = \frac{1}{4}(2x_k + 3) < \frac{1}{4}(4+3) = \frac{7}{4} < 2.$$

   Therefore, $x_n < 2 \ \forall n \in \mathbb{N}$ by induction.
   Claim: $(x_n)_{n\in\mathbb{N}}$ is strictly increasing.
   If $x_k < x_{k+1}$, then

$$x_{k+1} = \frac{1}{4}(2x_k + 3) < \frac{1}{4}(2x_{k+1} + 3) = x_{k+2}.$$

   By induction, $(x_n)_{n\in\mathbb{N}}$ is strictly increasing.
   By Theorem 3.7, $x_n \to x$ for some $x \in \mathbb{R}$. So

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{4}(2x_n + 3) = \frac{1}{4}(2x+3).$$

   Solving for x we get $x = \frac{3}{2}$.

2. The first three terms are $3, \frac{5}{3}, \frac{7}{5}$. Claim: $x_n > 1 \ \forall n \in \mathbb{N}$. If $x_k > 1$, then

$$0 < \frac{1}{x_k} < 1 \Rightarrow -\frac{1}{x_k} > -1 \Rightarrow x_{k+1} = 2 - \frac{1}{x_k} > 2 - 1 = 1.$$

   Therefore, $x_n > 1 \ \forall n \in \mathbb{N}$ by induction.
   Claim: $(x_n)_{n\in\mathbb{N}}$ is strictly decreasing. Fix $n \in \mathbb{N}$. Since $x_n^2 - 2x_n + 1 =$

$(x_n - 1)^2 > 0$ and $x_n > 0$, $x_n - 2 + \dfrac{1}{x_n} > 0$. Therefore, $x_n > 2 - \dfrac{1}{x_n} = x_{n+1}$. (This can also be done by induction.) By Theorem 3.7, $x_n \to x$ for some $x \in \mathbb{R}$. By Theorem 3.3, $x \geq 1$. So

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(2 - \frac{1}{x_n}\right) = 2 - \frac{1}{x}.$$

Solving for x we get $x = 1$.

3. The first three terms are $\sqrt{2}$, $\sqrt{2+\sqrt{2}} \cong 1.85$, $\sqrt{2+\sqrt{2+\sqrt{2}}} \cong 1.96$. <u>Claim</u>: $x_n < 2$ $\forall n \in \mathbb{N}$. If $x_k < 2$, then

$$x_{k+1} = \sqrt{2 + x_k} < \sqrt{4} = 2.$$

By induction, $x_n < 2$ $\forall n \in \mathbb{N}$. <u>Claim</u>: $(x_n)_{n\in\mathbb{N}}$ is strictly increasing. If $x_k < x_{k+1}$, then

$$x_{k+1} = \sqrt{2 + x_k} < \sqrt{2 + x_{k+1}} = x_{k+2}.$$

Therefore, $(x_n)_{n\in\mathbb{N}}$ is strictly increasing by induction. By Theorem 3.7, $x_n \to x$ for some $x \in \mathbb{R}$. So

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n} = \sqrt{2 + x}.$$

Therefore, $x^2 - x - 2 = 0$ and so $x = 2$ or $x = -1$. Since $x_n \geq \sqrt{2}$ $\forall n \in \mathbb{N}$, $x = 2$.

4. $x_{n+1} - x_n = \frac{1}{n+1} > 0 \Rightarrow (x_n)_{n\in\mathbb{N}}$ is strictly increasing. To see that $(x_n)_{n\in\mathbb{N}}$ is unbounded above and hence nonconvergent, use either (1) x_n is the nth partial sum of the divergent harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ or (2) write

$$\begin{aligned} x_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right) \\ &= 1 + \frac{n}{2}. \end{aligned}$$

(Of course, (2) is one proof that the harmonic series diverges.)

5. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded monotone decreasing sequence in $\mathbb{R}$. By Proposition 2.4, $\beta = \inf\{x_n : n \in \mathbb{N}\} \in \mathbb{R}$. Claim: $x_n \to \beta$. Let $\varepsilon > 0$. By Proposition 2.5, $\exists n_0 \in \mathbb{N}$ such that $x_{n_0} < \beta + \varepsilon$. Let $n \geq n_0$. Since $(x_n)_{n\in\mathbb{N}}$ is monotone decreasing, $\beta \leq x_n \leq x_{n_0} < \beta + \varepsilon$ and so $x_n \to \beta$. (Alternatively, one could use the fact that $(-x_n)_{n\in\mathbb{N}}$ is a bounded monotone increasing sequence and Exercise 2.2.5.)

6. $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}] = \emptyset$ and $\bigcap_{n=1}^{\infty}[1 - \frac{1}{n}, 1) = \emptyset$.

7. $\bigcap_{n=1}^{\infty}(n, \infty) = \bigcap_{n=1}^{\infty}[n, \infty) = \bigcap_{n=1}^{\infty}(-\infty, -n) = \bigcap_{n=1}^{\infty}(-\infty, -n] = \emptyset$.

8. (a) First note that $(b_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence which is bounded below by a_1. By Theorem 3.7 $\exists\ \beta \in \mathbb{R}$ such that $b_n \to \beta$.

   To show: $\beta \in \displaystyle\bigcap_{n=1}^{\infty} I_n$.

   By the proof of Theorem 3.7, $\beta = \inf\{b_n : n \in \mathbb{N}\}$ and so $\beta \leq b_n$ $\forall\ n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. We wish to show that $a_n \leq \beta$. If $k \leq n$, then $a_n \leq b_n \leq b_k$ (since the b_k's are monotone decreasing) and if $k > n$, $a_n \leq a_k \leq b_k$ (since the a_k's are monotone increasing). Therefore, a_n is a lower bound of $\{b_k : k \in \mathbb{N}\}$. Since β is the greatest lower bound of $\{b_k : k \in \mathbb{N}\}$, $a_n \leq \beta$. Since $n \in \mathbb{N}$ is arbitrary, $\beta \in [a_n, b_n]$ $\forall n \in \mathbb{N}$.

   (b) Let $x \in \displaystyle\bigcap_{n=1}^{\infty} I_n$. Then $a_n \leq x \leq b_n$ $\forall n \in \mathbb{N}$. Therefore, x is an upper bound of $\{a_n : n \in \mathbb{N}\}$ and x is a lower bound of $\{b_n : n \in \mathbb{N}\}$. Since $\alpha = \sup\{a_n : n \in \mathbb{N}\}$, $\alpha \leq x$; and since $\beta = \inf\{b_n : n \in \mathbb{N}\}$, $x \leq \beta$. So $\alpha \leq x \leq \beta$ and $\displaystyle\bigcap_{n=1}^{\infty} I_n \subset [\alpha, \beta]$.

   Let $x \in [\alpha, \beta]$. Then $a_n \leq \alpha \leq x \leq \beta \leq b_n$ $\forall n \in \mathbb{N}$, and so $x \in [a_n, b_n] = I_n$ $\forall n \in \mathbb{N}$. Therefore, $[\alpha, \beta] \subset \displaystyle\bigcap_{n=1}^{\infty} I_n$.

   (c) Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty}(b_n - a_n) = 0$, $\exists\ n_0 \in \mathbb{N}$ with $b_{n_0} - a_{n_0} < \varepsilon$. Then $0 \leq \beta - \alpha \leq b_{n_0} - a_{n_0} < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\beta - \alpha = 0$ and so $\alpha = \beta$. Therefore, $\displaystyle\bigcap_{n=1}^{\infty} I_n = \{\alpha\}$.

9. $\alpha = \sup A \in \mathbb{R}$ by the Completeness Axiom for $\mathbb{R}$.

Case 1: $\alpha \in A$. Let $x_n = \alpha$ $\forall n \in \mathbb{N}$.

Case 2: $\alpha \in \mathbb{R} \setminus A$. $\forall n \in \mathbb{N}$, by Proposition 2.5, $\exists\, x_n \in A$ with $\alpha - \frac{1}{n} < x_n < \alpha$. Therefore, $(x_n)_{n\in\mathbb{N}}$ is a sequence in A and $x_n \to \alpha$. By Theorem 3.9, $(x_n)_{n\in\mathbb{N}}$ has a monotone subsequence $(y_n)_{n\in\mathbb{N}}$ which, by Theorem 3.6, must also converge to α. If $(y_n)_{n\in\mathbb{N}}$ were monotone decreasing, then $\lim_{n\to\infty} y_n \leq y_1 < \alpha$ and hence $(y_n)_{n\in\mathbb{N}}$ is monotone increasing.

$\beta = \inf A \in \mathbb{R}$ by Proposition 2.4.

Case 1: $\beta \in A$. Let $x_n = \beta\ \forall n \in \mathbb{N}$.

Case 2: $\beta \in \mathbb{R} \setminus A$. $\forall n \in \mathbb{N}$, by Proposition 2.5, $\exists\, x_n \in A$ with $\beta < x_n < \beta + \frac{1}{n}$. Therefore, $(x_n)_{n\in\mathbb{N}}$ is a sequence in A and $x_n \to \beta$. By Theorem 3.9, $(x_n)_{n\in\mathbb{N}}$ has a monotone subsequence $(y_n)_{n\in\mathbb{N}}$ which, by Theorem 3.6, must also converge to β. If $(y_n)_{n\in\mathbb{N}}$ were monotone increasing, then $\lim_{n\to\infty} y_n \geq y_1 > \beta$ and hence $(y_n)_{n\in\mathbb{N}}$ is monotone decreasing.

3.5 Bolzano-Weierstrass Theorems

1. $[0, 7]$, by Theorem 3.3.

2. Theorem 3.10 implies that $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(y_n)_{n\in\mathbb{N}}$ which converges to some x in $\mathbb{R}$. Letting $\varepsilon = \frac{1}{2}$ in Proposition 3.1, $\exists\, n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow |y_n - x| < \frac{1}{2}$. Then $n \geq n_0$ and $m \geq n_0$ imply

$$|y_n - y_m| \leq |x - y_n| + |y_m - x| < \frac{1}{2} + \frac{1}{2} = 1.$$

   Since y_n and y_m are both integers, $y_n = y_m\ \forall\ n$ and m beyond n_0. Therefore, $(y_n)_{n\in\mathbb{N}}$ is eventually constant, and this constant must be x.

3. If $(x_n)_{n\in\mathbb{N}}$ is bounded, then $(x_n)_{n\in\mathbb{N}}$ has a subsequence which converges to some $L \in \mathbb{R}$ by Theorem 3.10. Since $(x_n)_{n\in\mathbb{N}}$ does not converge to L, $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(y_n)_{n\in\mathbb{N}}$ which is bounded away from L by Proposition 3.3. That is, $\exists\, \varepsilon > 0$ such that $|y_n - L| \geq \varepsilon\ \forall n \in \mathbb{N}$. Again

by Theorem 3.10, $(y_n)_{n\in\mathbb{N}}$ (and hence $(x_n)_{n\in\mathbb{N}}$) has a subsequence $(z_n)_{n\in\mathbb{N}}$ which converges to some $M \in \mathbb{R}$. Since $|z_n - L| \geq \varepsilon\ \forall n \in \mathbb{N}$, $M \neq L$. Thus, M and L are 2 distinct subsequential limits of $(x_n)_{n\in\mathbb{N}}$.

4. $\mathbb{R}$ since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$ (Exercise 2.3.4).

5. $\{\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{1 + \frac{1}{n}\}_{n\in\mathbb{N}} \cup \{2 + \frac{1}{n}\}_{n\in\mathbb{N}}$ has 0,1, and 2 as accumulation points.

6. We have $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence of distinct points in A with $x_n \to x$, and let U be a neighborhood of x. Since $(x_n)_{n\in\mathbb{N}}$ is eventually in U, $U \cap A$ is infinite, and so x is an accumulation point of A by Proposition 3.4.
   Now suppose that x is an accumulation point of A. By Proposition 3.4, $\left(x - \dfrac{1}{n}, x + \dfrac{1}{n}\right) \cap A$ is infinite $\forall n \in \mathbb{N}$. Choose $x_1 \in (x-1, x+1) \cap A$; choose $x_2 \in \left(x - \dfrac{1}{2}, x + \dfrac{1}{2}\right) \cap A$ with $x_2 \neq x_1$; choose $x_3 \in \left(x - \dfrac{1}{3}, x + \dfrac{1}{3}\right) \cap A$ with $x_3 \notin \{x_1, x_2\}$; etc. Then $(x_n)_{n\in\mathbb{N}}$ is a sequence of distinct points in A and $x_n \to x$.

7. Since x is an isolated point of A, $\exists$ a neighborhood V of x such that $V \cap A = \{x\}$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in A which converges to x, then $(x_n)_{n\in\mathbb{N}}$ must eventually be in $V \cap A$. Therefore, $(x_n)_{n\in\mathbb{N}}$ must eventually be the constant x.

8. Let $(x_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{Q}$ which converges to $\sqrt{2}$. (Such a sequence exists since $\mathbb{Q}$ is dense in $\mathbb{R}$.) Then $\{x_n : n \in \mathbb{N}\}$ is a bounded infinite subset of $\mathbb{Q}$ whose only accumulation point is $\sqrt{2}$.

3.6 Cauchy Sequences

1. (a) $\left(\frac{1}{n}\right)_{n=2}^{\infty}$, or any sequence in $(0,1)$ which converges to 0 or 1.
   (b) By Theorem 3.12, a Cauchy sequence in $[0,1]$ must converge to a real number, say x. By Theorem 3.3, $x \in [0,1]$.

2. (a) By Exercise 3.4.4, $(x_n)_{n\in\mathbb{N}}$ does not converge. By Theorem 3.12, $(x_n)_{n\in\mathbb{N}}$ is not Cauchy.
   (b) $|x_{n+1} - x_n| = \dfrac{1}{n+1} \to 0$.